Continued Fraction Sequences and Weave Design, Part 1: Introduction

Continued fractions are part of the "lost mathematics," the mathematics now considered too advanced for high school and too elementary for college.

　　— Petr Beckmann, *A History of Pi*

Introduction

Most persons taking courses in mathematics do not encounter continue fractions. When first encountered, they have a forbidding appearance. Yet continued fractions have an elegant theory and are important in several branches of mathematics.

The interest here is in sequences related to continued fractions, which can serve as the basis for weave design.

A continued fraction is a fraction in which the numerators and denominators may contain (continued) fractions. Displayed in their full laddered form, they look like this:

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \dots}}}}}}}$$

See Figure 1 on the next page for other examples.

The numerators and denominators in a continued fraction can themselves be complicated, as evidenced by Figure 1i. Most work on continued fractions deals with *ordinary* continued fractions, in which the numerators and denominators are numbers:

$$a_1 + \cfrac{b_1}{a_2 + \cfrac{b_2}{a_3 + \cfrac{b_3}{a_4 + \cfrac{b_4}{a_5 + \cfrac{b_5}{a_6 + \cfrac{b_6}{a_7 + \cfrac{b_7}{a_8 + \dots}}}}}}}$$

Two sequences completely characterize an ordinary continued fraction: $a_1, a_2, a_3, a_4, \dots$ and $b_1, b_2, b_3, b_4 \dots$.

A *simple* continued fraction is an ordinary continued fraction in which all the numerators are 1 and all the denominators are integers and positive except possibly a_1:

$$a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cfrac{1}{a_5 + \cfrac{1}{a_6 + \cfrac{1}{a_7 + \cfrac{1}{a_8 + \dots}}}}}}}$$

Only one sequence is needed to characterize a simple continued fraction. For example, the continued-fraction sequence for π is

3, 7, 15, 1, 292, 1, 1, 1, … .

As you'd expect, this sequence is infinite.

We'll stick to simple continued fractions in our explorations here.

There are several important facts about simple continued-fraction sequences:

1. Rational numbers (fractions) have finite sequences. An example is $11/13$, which has the sequence 0, 1, 5, 2.

2. Irrational numbers have infinite sequences.

3. Quadratic irrationals have periodic sequences. An example is $\sqrt{7}$, which has the sequence $2, \overline{1,1,1,4}$.

4. All other irrational numbers have non-periodic sequences. The sequence for π, shown above, is an example.

5. There is a one-to-one correspondence between an irrational number and its simple continued-fraction sequence. Furthermore, any periodic sequence of positive integers represents a unique irrational number. (For rational numbers, there are two equivalent sequences: one that ends … a_m, 1 and one that ends … $a_m - 1$.)

Computing Continued Fractions

Continued fractions are closely related to the familiar Euclidean algorithm for computing the greatest common divisor of two integers, i and j. Euclid's algorithm might look like this in pseudo-code:

```
until j = 0 do {
```

a
$$e - 1 = 1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{6 + \ldots}}}}}}}}$$

b
$$\frac{1}{e-2} = 1 + \cfrac{1}{2 + \cfrac{2}{3 + \cfrac{3}{4 + \cfrac{4}{5 + \cfrac{5}{6 + \cfrac{6}{7 + \cfrac{7}{8 + \ldots}}}}}}}$$

c
$$\frac{\pi}{2} = 1 - \cfrac{1}{3 - \cfrac{2\cdot3}{1 - \cfrac{1\cdot2}{3 - \cfrac{4\cdot5}{1 - \cfrac{3\cdot4}{3 - \cfrac{6\cdot7}{1 - \cfrac{5\cdot6}{3 - \ldots}}}}}}}$$

d
$$\frac{4}{\pi} = 1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cfrac{7^2}{2 + \cfrac{9^2}{2 + \cfrac{11^2}{2 + \cfrac{13^2}{2 + \ldots}}}}}}}$$

e
$$\sin(x) = \cfrac{x}{1 + \cfrac{x^2}{(2\cdot3 - x^2) + \cfrac{2\cdot3x^2}{(4\cdot5 - x^2) + \cfrac{4\cdot5x^2}{(6\cdot7 - x^2) + \ldots}}}}$$

f
$$\tan(1) = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{5 + \cfrac{1}{1 + \cfrac{1}{7 + \cfrac{1}{1 + \ldots}}}}}}}}$$

g
$$\frac{1+\sqrt{5}}{2} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ldots}}}}}}$$

h
$$\log(1+x) = \cfrac{x}{1 + \cfrac{1^2 x}{2 + \cfrac{1^2 x}{3 + \cfrac{2^2 x}{4 + \cfrac{2^2 x}{5 + \cfrac{3^2 x}{6 + \cfrac{3^2 x}{7 + \ldots}}}}}}}$$

i
$$\sqrt[3]{2} + 2 = \cfrac{1}{3 + \cfrac{\cfrac{1}{3 + \cfrac{3 + \ldots}{3 + \ldots}}}{3 + \cfrac{\cfrac{1}{3 + \cfrac{3 + \ldots}{3 + \ldots}}}{3 + \cfrac{\cfrac{1}{3 + \cfrac{3 + \ldots}{3 + \ldots}}}{3 + \cfrac{\cfrac{1}{3 + \cfrac{3 + \ldots}{3 + \ldots}}}{3 + \cfrac{\cfrac{1}{3 + \cfrac{3 + \ldots}{3 + \ldots}}}{3 + \ldots}}}}}}$$

Figure 1. A Gallery of Continued Fractions

```
    r := remdr(i, j)
    i := j
    j := r
    }

  print(i)          # previous value of j
```

The terms in the simple continued faction for i / j consist of values of $i \div j$ (integer division, remainder discarded) in the loop above:

```
  until j = 0 do {
  print(i ÷ j)
  r := remdr(i, j)
  i := j
  j := r
  }
```

The problem with trying to compute continued fractions for irrational numbers is that floating-point numbers are finite approximations to real numbers, and hence they really are rational numbers whose values are "close" to the corresponding real numbers. For example, the standard 64-bit floating-point encoding for π is

$$7074237752028440 / 2^{51}$$

The corresponding continued-fraction sequence is, of course, finite:

$$3, 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 3, 3, 2,$$
$$1, 3, 3, 7, 2, 1, 1, 3, 2, 42, 2$$

and only the first 13 terms are the same as for the sequence for the actual irrational number:

$$3, 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1,$$
$$2, 2, 2, 2, 1, 84, 2, 1, \ldots$$

Patterns

Simple continued-fraction sequences for rational numbers usually are short and any patterns are accidental and mostly uninteresting.

Since quadratic irrationals have periodic simple continued-fraction sequences, they have patterns that may be of interest in designing weaves.

Simple continued-fraction sequences for other irrationals are not periodic and most have no evident patterns.

Some, however, do. An example is $\tan(1)$ (see Figure 1f), whose simple continued-fraction sequence is

$$1, \overline{1, 2n+1} \qquad\qquad n = 1, 2, 3, \ldots$$

Another example is $e - 1$ (see Figure 1a), whose simple continued-fraction sequence is

$$1, \overline{1, 2n, 1} \qquad\qquad n = 1, 2, 3, \ldots$$

Such sequences have periodic *forms*. The simple continued-fraction sequence for π has no such structure, but there is an ordinary continued-fraction for $\pi/4$ (see Figure 1d) that has numerator and denominator sequences with periodic forms:

numerators: $\quad \overline{(2n-1)^2} \quad n = 1, 2, 3, \ldots$

denominators: $\quad 1, \overline{2}$

Sequences with periodic forms are on our agenda.

So far, we've not said anything about the use of continued fractions in designing weaves. We'll get to that in a subsequent paper. Figures 2 through 4 indicate some possibilities.
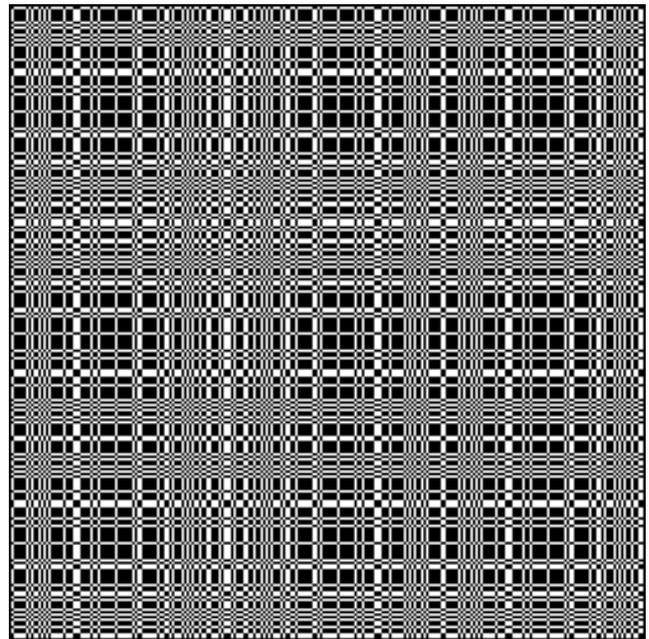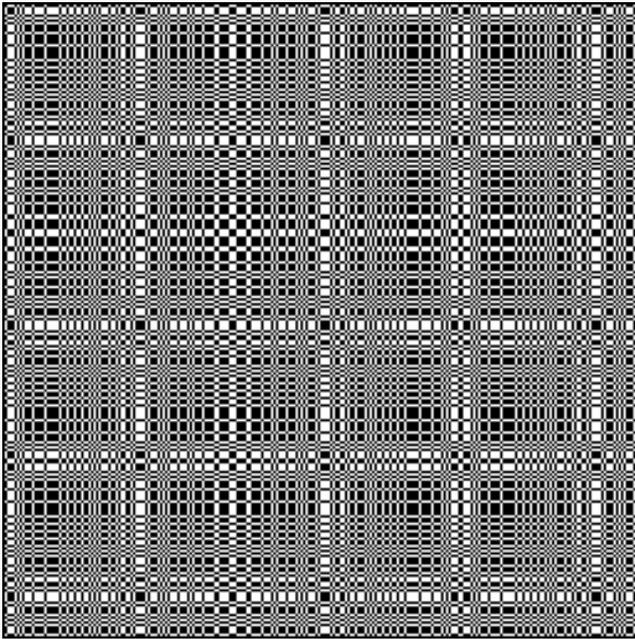

Figure 2. $\sqrt{10089}$ **Tabby**
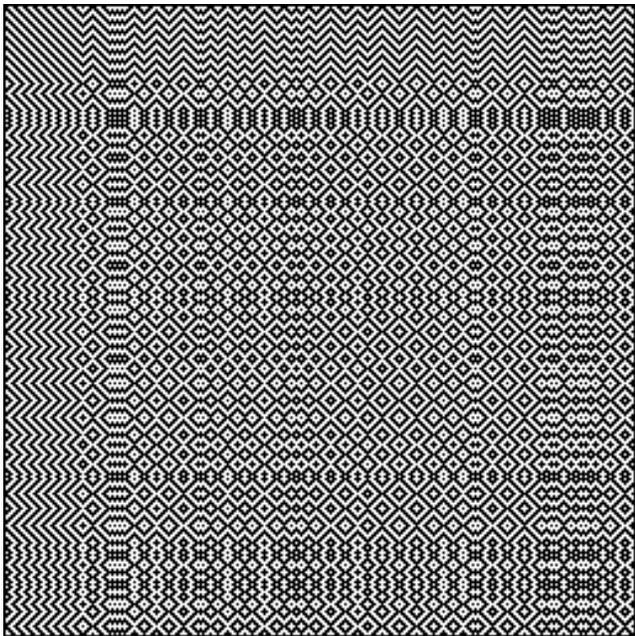
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Figure 3. $\sqrt{9949}$ **Tabby**

Figure 4. $\sqrt{9949}$ **Twill**

Learning More About Continued Fractions

Much of the literature about continued fractions is highly technical and specialized. There are, however, a few books that are accessible [1-3]. There also are Web resources [4-5].

More to Come

As mentioned earlier, simple continued-fraction sequences for quadratic irrationals are periodic. This will be the topic of a subsequent article.

References

1. *Continued Fractions*, C. D. Olds, Mathematical Association of America, 1963.

2. *The Higher Arithmetic*, H. Davenport, Cambridge University Press, 1999.

3. *What is Mathematics?*, Richard Courant and Herbert Robbins, Oxford University Press, 1996.

4. N umber Theory: Continued Fractions:
 http://mathworld.wolfram.com/topics/ContinuedFractions.html

5. Continued Fractions ... an Introduction:
 http://archives.math.utk.edu/articles/atuyl/confrac/

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